POLITECNICO DI MILANO

Transient analysis of CTMCs

Reliability and CTMCs

When using CTMCs for computing the Reliability of a system, special care should be taken.

In particular, even if the state space can be computed using the techniques seen in the previous lessons, the transitions must be defined in a special way.

Moreover, as a difference with respect to the computation of Availability using CTMCs, the structure of the CTMC not only depends on the components, but also on how they are connected (e.g. in series or in parallel).

Matrix exponential

We have seen that the transient solution of a CTMC can be computed solving the following ODE:

$$\frac{d\pi(t)}{dt} = \pi(t) \cdot Q$$

The *Matrix Exponential* of a matrix Q is defined as:

$$e^{M} = \sum_{k=0}^{\infty} \frac{M^k}{k!} \qquad e^{Qt} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!}$$

It can be easily shown that:

$$\frac{d e^{Qt}}{dt} = \sum_{k=0}^{\infty} \frac{Q^k}{k!} \frac{dt^k}{dt} = \sum_{k=1}^{\infty} \frac{Q^k}{(k-1)!} t^{k-1} = Q e^{Qt} = e^{Qt} Q$$

Matrix exponential

The Matrix exponential is thus a solution of the Chapman-Kolmogorov equation, in particular, since $e^{Q*0}=\boldsymbol{I}$, we have:

$$\frac{d\pi(t)}{dt} = \pi(t) \cdot Q$$

$$\pi(t) = \pi(0) \cdot e^{Qt}$$

Several efficient algorithms exists to efficiently compute the matrix exponential.

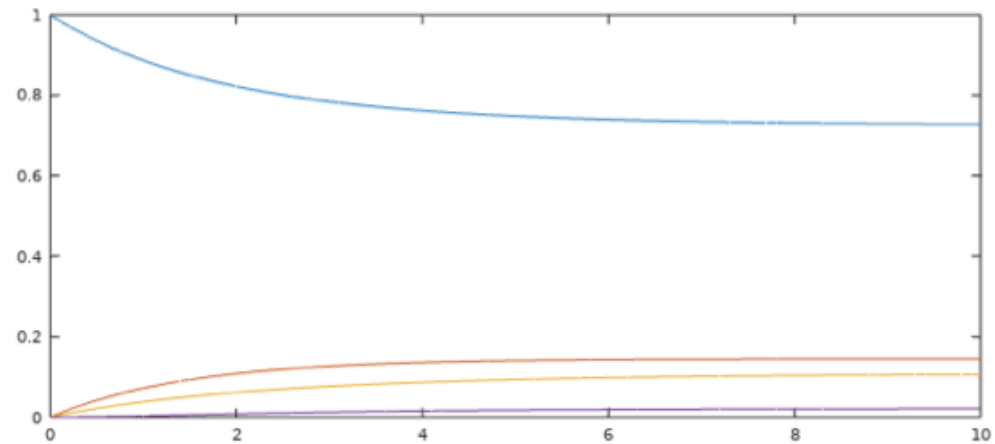
Most of mathematical packages includes functions for computing it: for example Octave and Matlab have a function called `expm(M)`.

Solution with Matrix exponential

```
MTTF1 = 10;
MTTF2 = 20;
MTTR1 = 2;
MTTR2 = 3;

l1 = 1/MTTF1;
l2 = 1/MTTF2;
m1 = 1/MTTR1;
m2 = 1/MTTR2;

Q = [-l1-l2,  l1   ,  l2   ,  0;
       m1  ,-m1-l2,   0   , l2;
       m2  ,   0   ,-m2-l1, l1;
        0  ,  m2   ,  m1   ,-m2-m1];

p0 = [1, 0, 0, 0];

t = linspace(0,10,101);
for i=1:size(t,2)
   Sol(:,i) = p0 * expm(Q * t(i));
end

plot(t, Sol, "-")
```

$$Q = \begin{vmatrix} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 & \\ \mu_1 & -\mu_1-\lambda_2 & & \lambda_2 \\ \mu_2 & & -\mu_2-\lambda_1 & \lambda_1 \\ & \mu_2 & \mu_1 & -\mu_2-\mu_1 \end{vmatrix}$$

$$\pi(0) = \begin{vmatrix} 1 & 0 & 0 & 0 \end{vmatrix} \qquad \frac{d\pi(t)}{dt} = \pi(t)\cdot Q$$



(in this case, the same code will work both in Octave and Matlab)

State classification

In a CTMC, a state can either be:

» Ergodic

» Transient

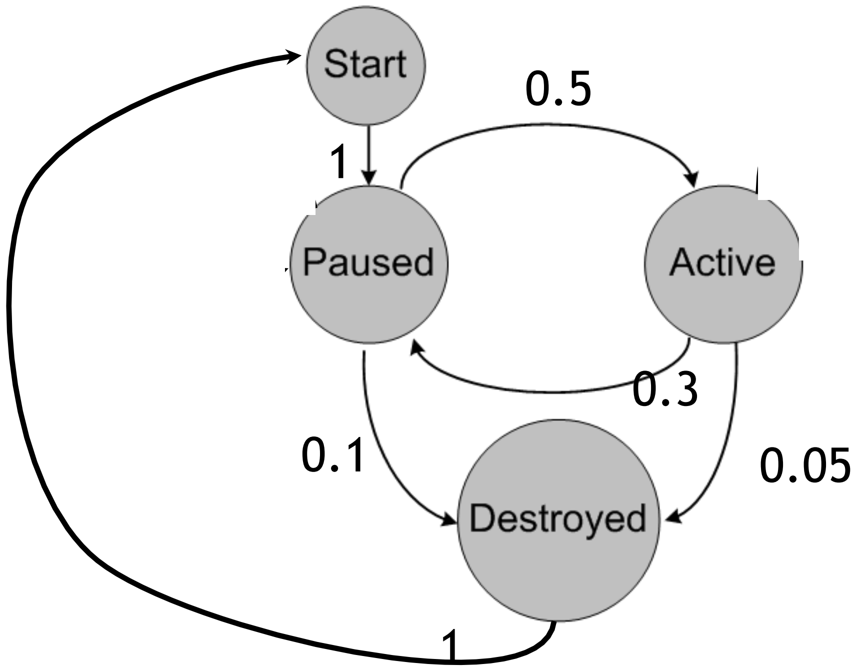
» Absorbing

Ergodic States

An **Erogodic** (or *recurrent*) state, is a state that can return indefinitely in time.

Ergodic are the “normal” states of a system.

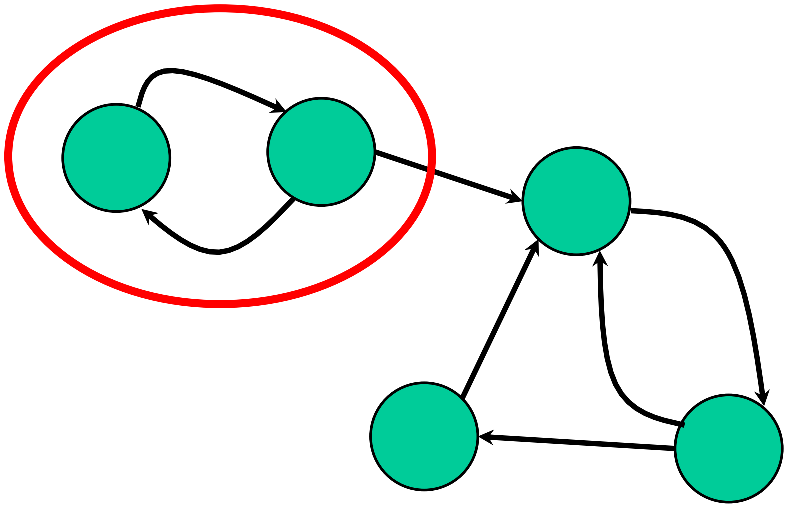
For ergodic states it is meaningful to compute the *steady-state* distribution.



Transient States

Transient states are the one for which their probability tends to zero (in finite chains) as the time goes to infinity.

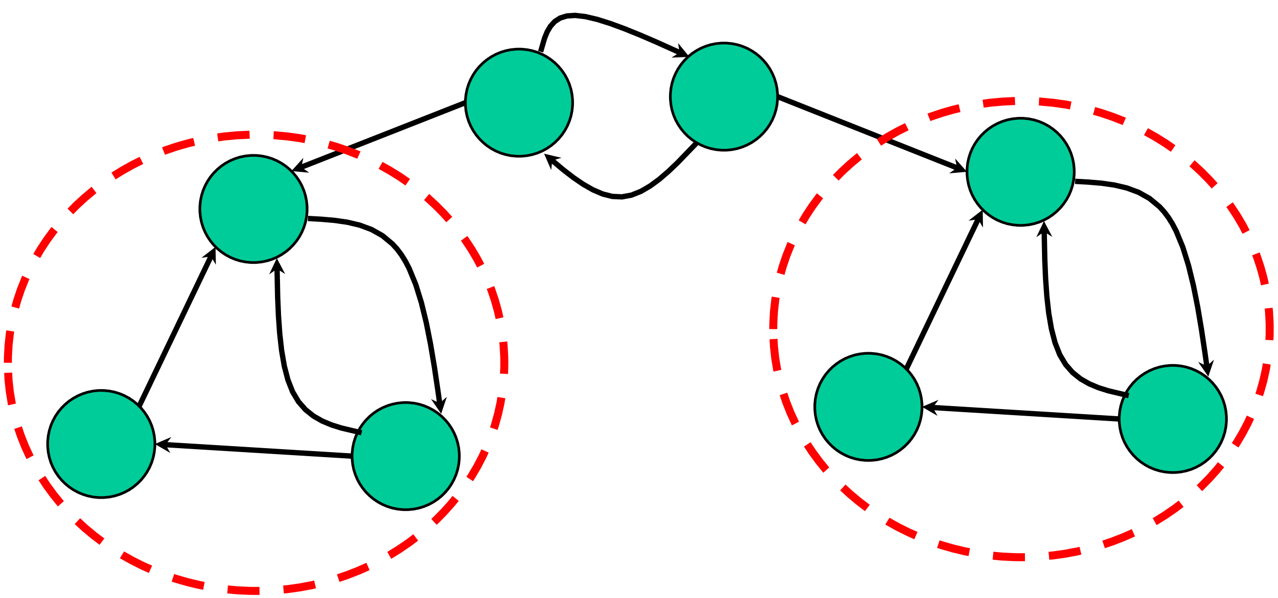
They are states that sooner or later will be abandoned by the system.



Transient States

If there are transient states, the chain may have more than one *Strong components*: that is a subset of the states where the system might be "trapped" inside.

If there are more than one strong components, the steady-state solution is not unique, and depends on the initial state of the model. Moreover it has a different meaning, since once a component has been chosen, the other will have zero probability.

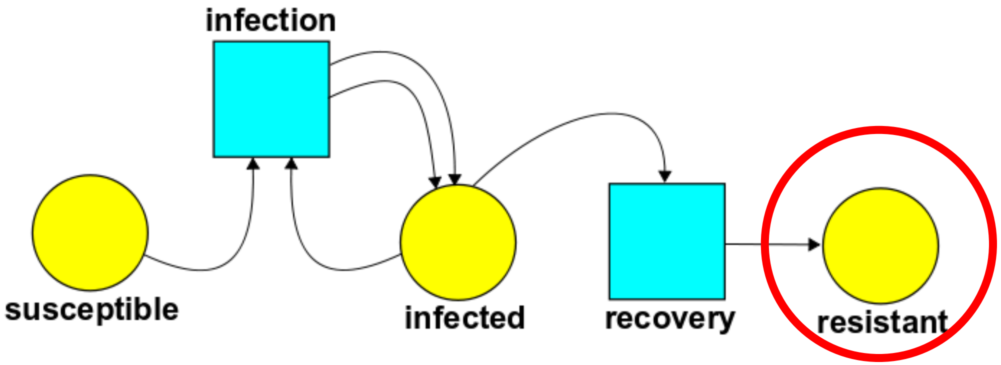


Absorbing States

A state in which there are no output transitions is called an **Absorbing** state.

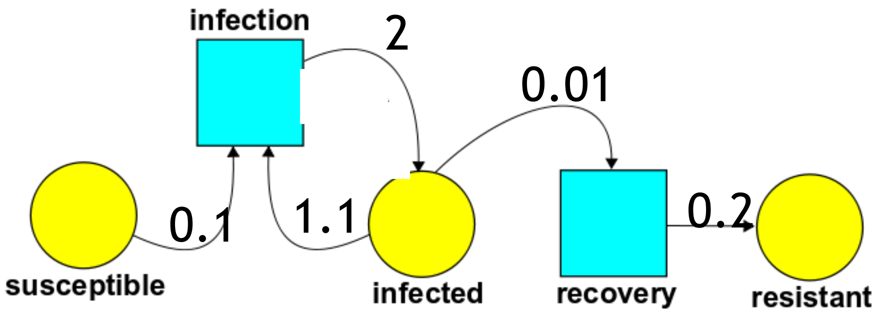
An absorbing state is a strong component, composed by a single state.

When the system reaches an absorbing state, it is trapped inside it forever.



Absorbing States

In the infinitesimal generator of a CTMC, the presence of absorbing states corresponds to lines entirely composed of zeros.



$$Q = \begin{vmatrix} -0.1 & 0.1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1.1 & -1.11 & 0.01 & 0 \\ 0 & 0 & 0 & -0.2 & 0.2 \\ \boxed{0 \quad 0 \quad 0 \quad 0 \quad 0} \end{vmatrix}$$

Absorbing States

If there is a single absorbing state, its steady-state probability is 1, and the steady-state probability of all the other states is 0.

- If there are more than one absorbing state, since they are strong components, their steady state distribution depends on the initial state of the model.

Even if their steady-state distribution is trivial, their transient probability can be used to study interesting properties of a system.

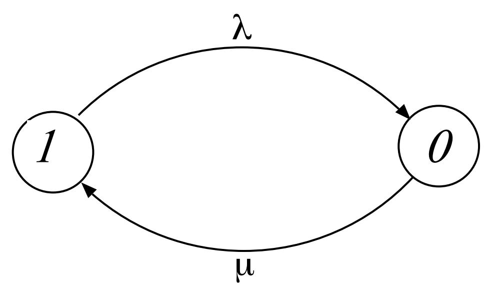
To compute *Reliability*, absorbing states must be used, and their distribution should be computed.

Reliability of a single component

We have seen for availability, that a single (repairable) component, can be modeled by a two state CTMC:

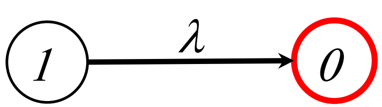
$$\lambda = \frac{1}{MTTF}$$

$$\mu = \frac{1}{MTTR}$$



$$Q = \begin{vmatrix} -\lambda & \lambda \\ \mu & -\mu \end{vmatrix}$$

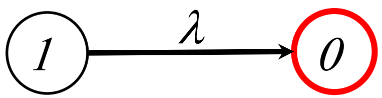
For reliability, we are not interested in considering the repair: we can thus remove the arc that returns on the up state.



$$Q = \begin{vmatrix} -\lambda & \lambda \\ 0 & 0 \end{vmatrix}$$

Reliability of a single component

The system enters the absorbing state when it fails, and never leaves it.



$$Q = \begin{vmatrix} -\lambda & \lambda \\ 0 & 0 \end{vmatrix}$$

The transient distribution of the absorbing state, corresponds to the transient probability of the system having failed.

By definition, this is the *Unreliability of the system*. The *Reliability* can then be obtained accordingly

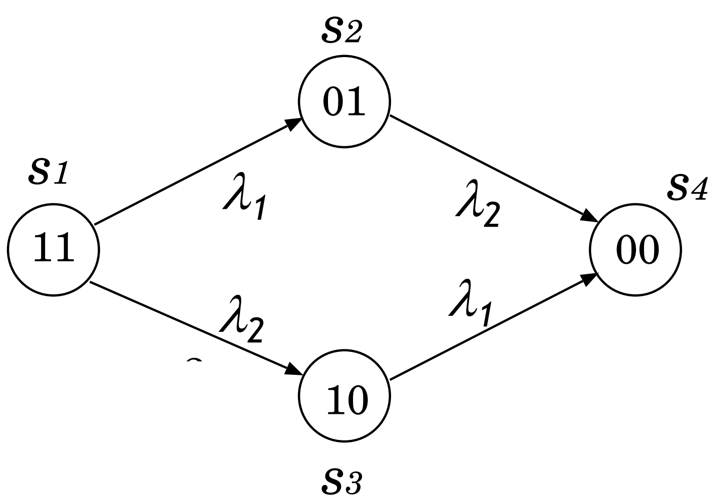
$$F(t) = \pi_0(t) \qquad R(t) = 1 - F(t) = 1 - \pi_0(t)$$

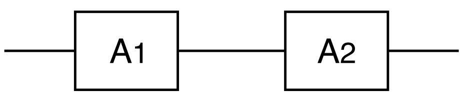
Two components case

Let us now consider a two component system, and compute its reliability

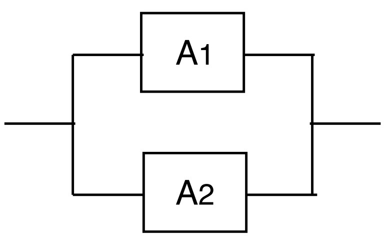
We have to make all the states where the system is down (Ω_d) absorbing by deleting all the outgoing transitions.

However this creates a different CTMC for each configuration.





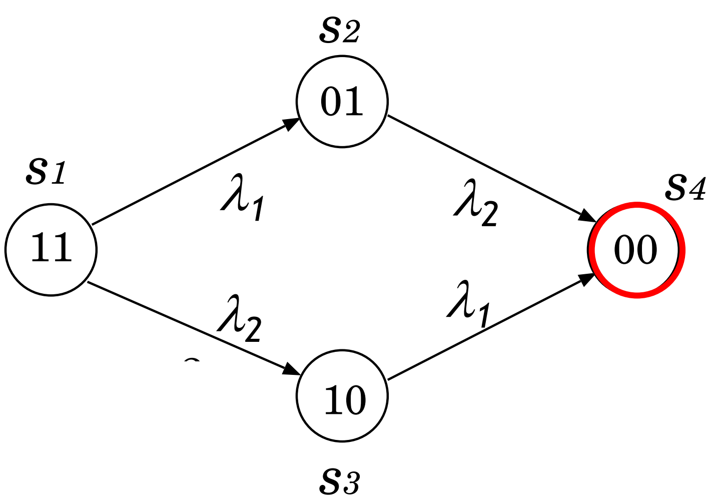
$\Omega_u = \{s_1\}$
$\Omega_d = \{s_2, s_3, s_4\}$

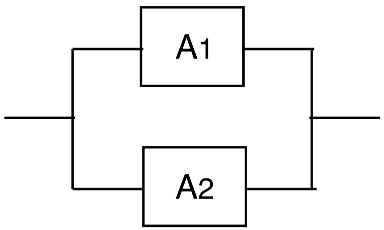


$\Omega_u = \{s_1, s_2, s_3\}$
$\Omega_d = \{s_4\}$

Two components case: parallel system

For the parallel system, the only state belonging to Ω_d is already absorbing:



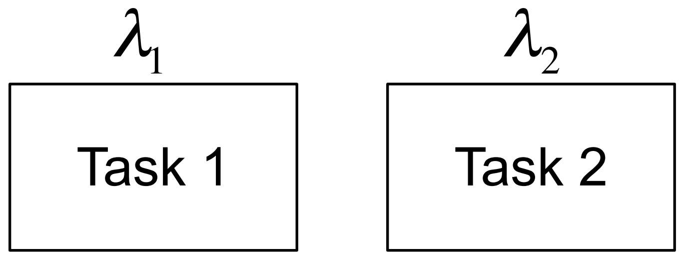


$\Omega_u = \{s_1, s_2, s_3\}$
$\Omega_d = \{s_4\}$

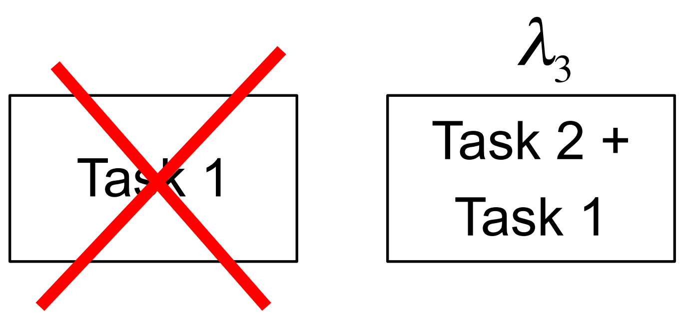
$$Q = \begin{vmatrix} -\lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 0 & -\lambda_2 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_1 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Example: change of failure rate (1)

Consider a dual core system: two tasks are run in parallel on the two cores. Since they use the CPU at different levels, the core running task 1 has an higher failure rate (λ_1) than the core running task 2 (failure rate λ_2).

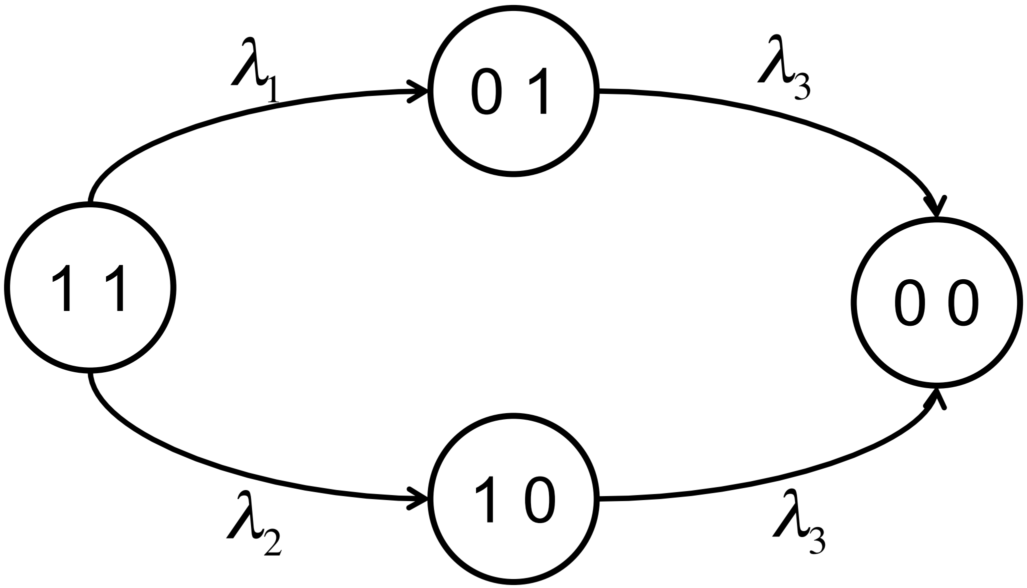


If one of the two cores fail, the surviving one will take care of both tasks. This however will lead to an increased failure rate λ_3



Example: change of failure rate (2)

The corresponding Markov chain becomes:

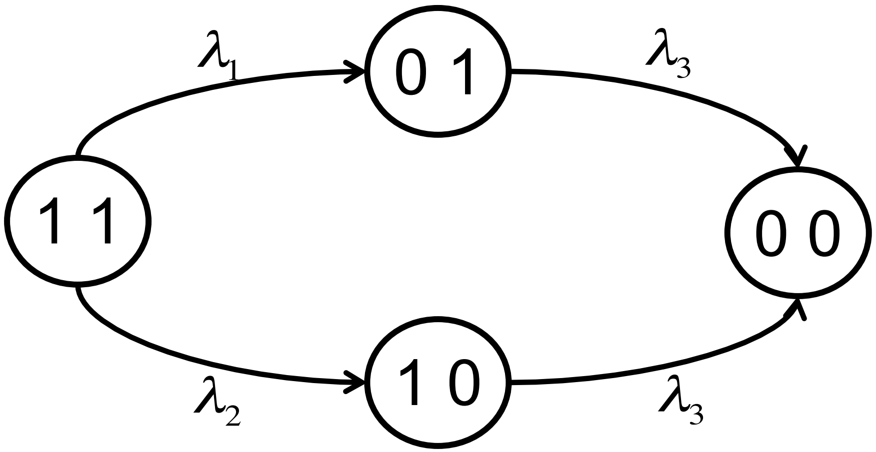


$$\lambda_3 > \lambda_1 > \lambda_2$$

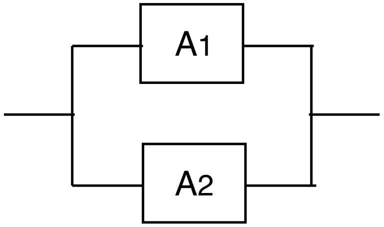
$$Q = \begin{vmatrix} -\lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 0 & -\lambda_3 & 0 & \lambda_3 \\ 0 & 0 & -\lambda_3 & \lambda_3 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Example: change of failure rate (3)

The two components are in parallel (even if the failure rate changes):



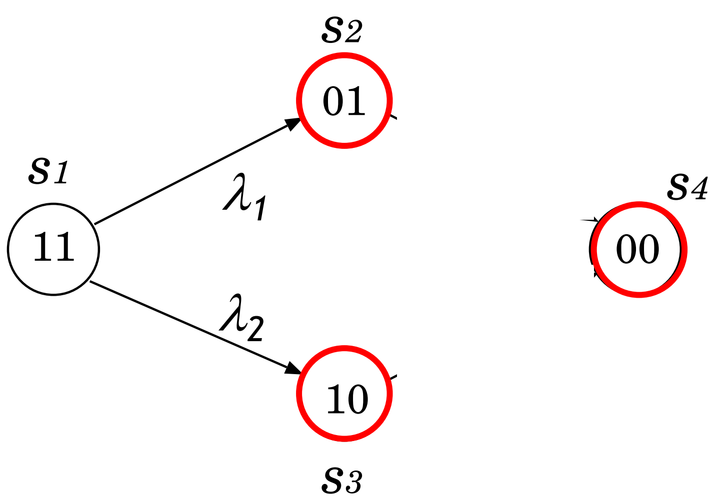
$$Q = \begin{vmatrix} -\lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 0 & -\lambda_3 & 0 & \lambda_3 \\ 0 & 0 & -\lambda_3 & \lambda_3 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

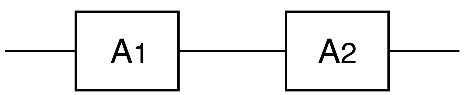


$\Omega_u = \{s_1,\ s_2,\ s_3\}$
$\Omega_d = \{s_4\}$

Two components case: serial system

For the series system instead we have:



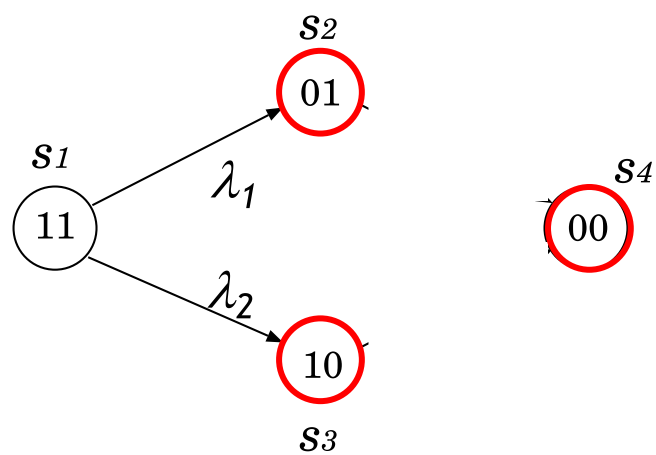


$\Omega_u = \{s_1\}$
$\Omega_d = \{s_2, s_3, s_4\}$

$$Q = \begin{vmatrix} -\lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Two components case: serial system

The series model has three absorbing states, corresponding to the states in which the system is not working:

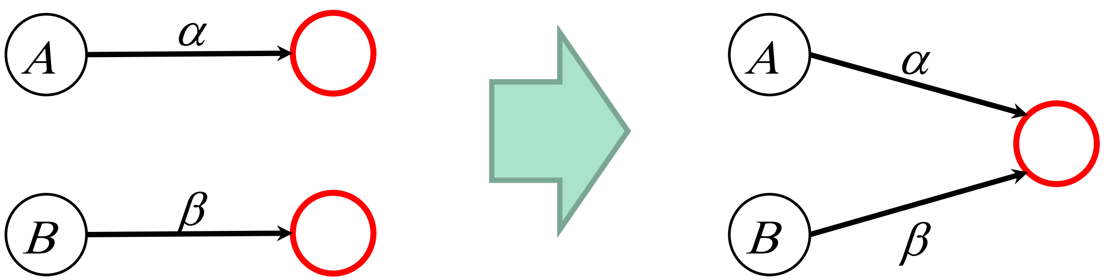


This may pose some difficulties, since the model is not ergodic, with several different strong components.

However, since all the failure states are equivalent and we are only interested in the sum of their probabilities, they can be grouped together in a single state.

Lumping

State grouping is called *lumping*: for absorbing states it can be done using two simple rules.

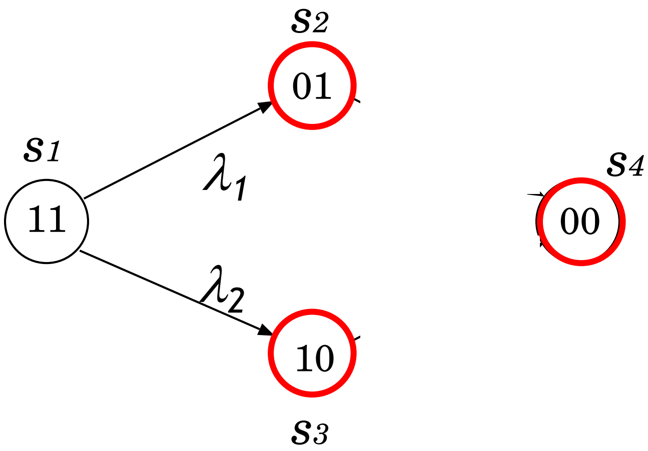


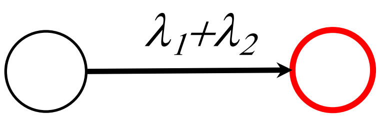
All the arcs that goes from a normal to an absorbing state, should be redirected to the *single absorbing* state (the *lumped* state).

Lumping

Then, if there are states from which several arrows are directed to the lumped state, they can be fused in a single arc. The rate of this new arc must correspond to the sum of the rates of the initial arcs.

Thus, after lumping, for the reliability of the series of two repairable components we have that:





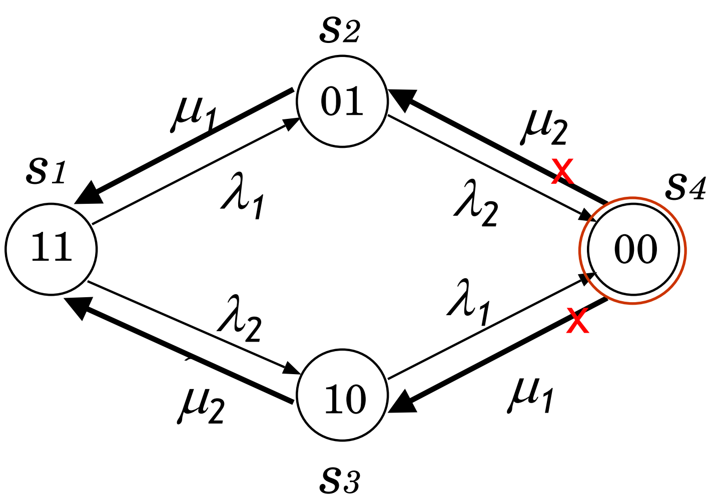
$$Q = \begin{vmatrix} -\lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \\ 0 & 0 \end{vmatrix}$$

Two components: reliability of a parallel repairable system

Let us now consider a two component repairable system, and compute the probability of uninterrupted service up to a time t.

This measure is relevant only for parallel systems, because for the serial case there is no difference with respect to the non-repairable case.

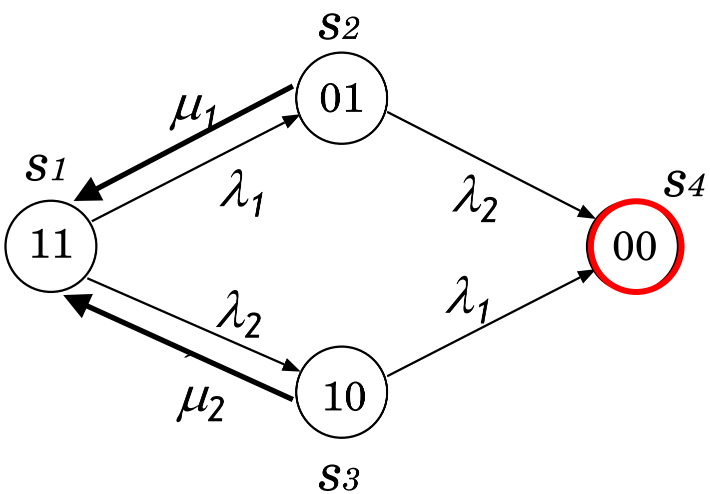
Again, we have to make the states where the system is not working (state s_4) absorbing to "accumulate" probability once failure occurs.

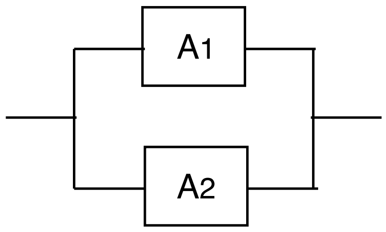


$\Omega_u = \{s_1, s_2, s_3\}$
$\Omega_d = \{s_4\}$

Two components case

We then have:





$\Omega_u = \{s_1, s_2, s_3\}$
$\Omega_d = \{s_4\}$

$$Q = \begin{vmatrix} -\lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ \mu_1 & -\lambda_2 - \mu_1 & 0 & \lambda_2 \\ \mu_2 & 0 & -\lambda_1 - \mu_2 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Distribution of the time to absorption

If we have a CTMC with a single absorbing state (let us imagine it is the last state), the distribution of the time to absorption *F(t)*, can be computed with the matrix exponential using only the sub-matrix **Q'** corresponding to non-absorbing states.

Q'

$$Q = \begin{vmatrix} -\lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 0 & -\lambda_2 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_1 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$F(t) = 1 - \pi'(0) \cdot e^{Q't} \cdot u$$

Distribution of the time to absorption

$$Q = \left| \begin{array}{cccc} -\lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 0 & -\lambda_2 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_1 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{array} \right|$$

Q'

$$F(t) = 1 - \pi'(0) \cdot e^{Q't} \cdot u$$

π'(0) represents the initial state distribution, excluding the absorbing state.

If we imagine that the systems starts with all its components working, and that this is the first state, we have *π'(0)=(1, 0, … , 0)* .

u = *(1, … , 1)^T* is a column vector, with all elements equal to 1.

In this setting the reliability *R(t)* of the system can be computed as:

$$R(t) = \pi'(0) \cdot e^{Q't} \cdot u$$

Mean time to absorption

Similarly, the sub-matrix **Q'** can be used to compute the mean time to absorption σ.

$$Q = \begin{array}{l} Q' \\ \end{array} \begin{vmatrix} -\lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 0 & -\lambda_2 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_1 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{vmatrix} \qquad \sigma = -\pi'(0) \cdot \left(Q'\right)^{-1} \cdot u$$

The mean time to failure (MTTF), can thus be considered equal to σ:

$$MTTF = \sigma = -\pi'(0) \cdot \left(Q'\right)^{-1} \cdot u$$

Example implementation

```
MTTF1 = 10;
MTTF2 = 20;
MTTR1 = 2;
MTTR2 = 3;

l1 = 1/MTTF1;
l2 = 1/MTTF2;
m1 = 1/MTTR1;
m2 = 1/MTTR2;

Qp = [-l1-l2,  l1 ,  l2;
         0   ,  -l2 ,   0;
         0   ,   0  , -l1];

p0p = [1, 0, 0];

t = linspace(0,100,101);
u = ones(3,1);
for i=1:size(t,2)
  R(1,i) = p0p * expm(Qp * t(i)) * u;
end

plot(t, R, "-")
MTTF = -p0p * inv(Qp) * u
```

Q'

$$Q = \begin{vmatrix} -\lambda_1 - \lambda_2 & \lambda_1 & \lambda_2 & 0 \\ 0 & -\lambda_2 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_1 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$\pi(0) = \begin{vmatrix} 1 & 0 & 0 & 0 \end{vmatrix}$$

π'(0)

(in this case, the same code will work both in Octave and Matlab)

Examples: discussion

We can solve numerically the models of the five cases just introduced (1 - series, 2 -parallel, 3 - parallel repairable, 4 - parallel with rate change, 5 – parallel repairable with rate change) for $\lambda_1=5$, $\lambda_2=3$, $\lambda_3=7$, $\mu_1=1$, $\mu_2=1.5$

$$Q_1 = \begin{vmatrix} -8 & 8 \\ 0 & 0 \end{vmatrix}$$

$$Q_2 = \begin{vmatrix} -8 & 5 & 3 & 0 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$Q_3 = \begin{vmatrix} -8 & 5 & 3 & 0 \\ 1 & -4 & 0 & 3 \\ 1.5 & 0 & -6.5 & 5 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$Q_4 = \begin{vmatrix} -8 & 5 & 3 & 0 \\ 0 & -7 & 0 & 7 \\ 0 & 0 & -7 & 7 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$Q_5 = \begin{vmatrix} -8 & 5 & 3 & 0 \\ 1 & -8 & 0 & 7 \\ 1.5 & 0 & -8.5 & 7 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Examples: discussion

1 – series

2 – parallel

3 - parallel repairable

4 - parallel with rate change

5 – parallel repairable with rate change

- for λ_1=5, λ_2=3, λ_3=7, μ_1=10, μ_2=15

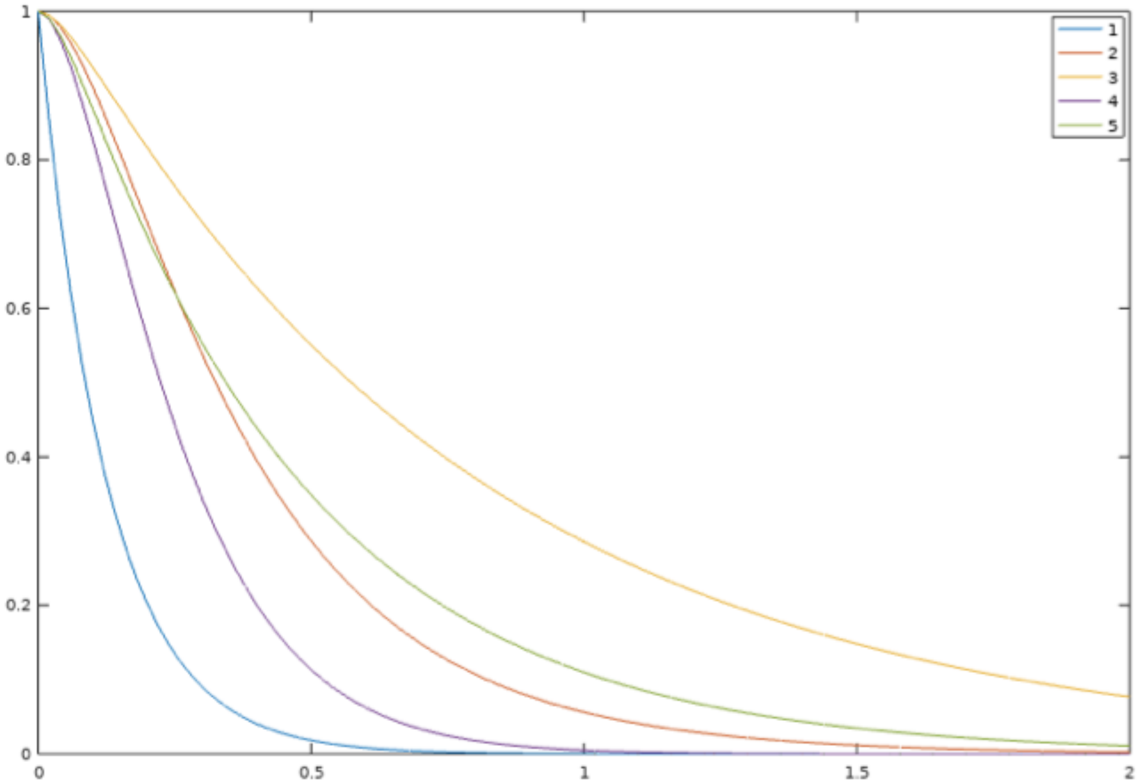
$MTTF_1$ = 0.12500

$MTTF_2$ = 0.40833

$MTTF_3$ = 0.80606

$MTTF_4$ = 0.26786

$MTTF_5$ = 0.47471



Since the repair rates are comparable with the failure rates, even if they increase the lifetime of the system, their effectiveness is limited.

Rate change reduces significantly the lifetime of the system.